

Zero dissipation limit to rarefaction wave with vacuum for 1-D compressible Navier-Stokes equations

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Abstract

It is well-known that one-dimensional isentropic gas dynamics has two elementary waves, i.e., shock wave and rarefaction wave. Among the two waves, only the rarefaction wave can be connected to vacuum. Given a rarefaction wave with one-side vacuum state to the compressible Euler equations, we can construct a sequence of solutions to one-dimensional compressible isentropic Navier-Stokes equations which converge to the above rarefaction wave with vacuum as the viscosity tends to zero. Moreover, the uniform convergence rate is obtained. The proof consists of a scaling argument and elementary energy analysis based on the underlying rarefaction wave structures.

Keywords: compressible Navier-Stokes equations, zero dissipation limit, rarefaction wave, vacuum.

1 Introduction and main result

In this paper, we investigate the zero dissipation limit of the one-dimensional compressible isentropic Navier-Stokes equations

$$\begin{cases} \rho_t + (\rho u)_x = 0, & x \in \mathbf{R}, t > 0, \\ (\rho u)_t + (\rho u^2 + p(\rho))_x = \epsilon u_{xx}, \end{cases} \quad (1.1)$$

where $\rho(t, x) \geq 0$, $u(t, x)$ and p represent the density, the velocity and the pressure of the gas, respectively and $\epsilon > 0$ is the viscosity coefficient. Here we assume that the viscosity coefficient ϵ is a positive constant and the pressure p is given by the γ -law:

$$p(\rho) = \frac{\rho^\gamma}{\gamma}$$

with $\gamma > 1$ being the gas constant.

Formally, as ϵ tends to zero, the limit system of the compressible Navier-Stokes equations (1.1) is the following inviscid Euler equations

$$\begin{cases} \rho_t + (\rho u)_x = 0, \\ (\rho u)_t + (\rho u^2 + p(\rho))_x = 0. \end{cases} \quad (1.2)$$

The Euler system (1.2) is a strictly hyperbolic one for $\rho > 0$ whose characteristic fields are both genuinely nonlinear, that is, in the equivalent system

$$\begin{pmatrix} \rho \\ u \end{pmatrix}_t + \begin{pmatrix} u & \rho \\ p'(\rho)/\rho & u \end{pmatrix} \begin{pmatrix} \rho \\ u \end{pmatrix}_x = 0,$$

the Jacobi matrix

$$\begin{pmatrix} u & \rho \\ p'(\rho)/\rho & u \end{pmatrix}$$

has two distinct eigenvalues

$$\lambda_1(\rho, u) = u - \sqrt{p'(\rho)}, \quad \lambda_2(\rho, u) = u + \sqrt{p'(\rho)}$$

with corresponding right eigenvectors

$$r_i(\rho, u) = (1, (-1)^i \frac{\sqrt{p'(\rho)}}{\rho})^t, \quad i = 1, 2,$$

such that

$$r_i(\rho, u) \cdot \nabla_{\rho, u} \lambda_i(\rho, u) = (-1)^i \frac{\rho p''(\rho) + 2p'(\rho)}{2\rho \sqrt{p'(\rho)}} \neq 0, \quad i = 1, 2.$$

We can define the i -Riemann invariant ($i = 1, 2$) by

$$\Sigma_i(\rho, u) = u + (-1)^{i+1} \int^\rho \frac{\sqrt{p'(s)}}{s} ds$$

such that

$$\nabla_{(\rho, u)} \Sigma_i(\rho, u) \cdot r_i(\rho, u) \equiv 0, \quad \forall \rho > 0, u.$$

The study of the limiting process of viscous flows when the viscosity tends to zero, is one of the important problems in the theory of the compressible fluid. When the solution of the inviscid flow is smooth, the zero dissipation limit can be solved by classical scaling method. However, the inviscid compressible flow contains singularities such as shock and the vacuum in general. Therefore, how to justify the zero dissipation limit to the Euler equations with basic wave patterns and/or the vacuum is a natural and difficult problem.

There have been many results on the zero dissipation limit of the compressible fluid with basic wave patterns without vacuum. For the system of the hyperbolic conservation laws with artificial viscosity

$$u_t + f(u)_x = \varepsilon u_{xx},$$

Goodman-Xin [5] first verified the viscous limit for piecewise smooth solutions separated by non-interacting shock waves using a matched asymptotic expansion method. Later Yu [22] proved it for the corresponding hyperbolic conservation laws with both shock and initial layers. In 2005, important progress made by Bianchini-Bressan[1] justifies the vanishing viscosity limit in BV space even though the problem is still unsolved for the physical system such as the compressible Navier-Stokes equations. For the compressible isentropic Navier-Stokes equations (1.1), Hoff-Liu [6] first proved the vanishing viscosity limit for piecewise constant shock even with initial layer. Later Xin [20] obtained the zero dissipation limit for rarefaction waves without vacuum for both rarefaction wave data and well-prepared smooth data. Then Wang [18] generalized the result of Goodman-Xin [5] to the isentropic Navier-Stokes equations (1.1). For the full Navier-Stokes equations where the conservation of the energy is also involved, there are also many results on the zero dissipation limit to the corresponding full Euler system with basic wave patterns without vacuum. We refer to Jiang-Ni-Sun [11] and Xin-Zeng [21] for the rarefaction wave, Wang [19] for the shock wave, Ma [14] for the contact discontinuity and Huang-Wang-Yang [9, 10] for the superposition of two rarefaction waves and a contact discontinuity and the superposition of one shock and one rarefaction wave cases.

More recently, Chen-Perepelitsa [3] proved the vanishing viscosity to the compressible Euler equations for the compressible Navier-Stokes equations (1.1) by compensated compactness method for the general case if the far field of the initial values of Euler system (1.2) has no vacuums. Note that this result is quite universal since the initial values of the Euler system can contain vacuum states in the interior domain. Huang-Pan-Wang-Wang-Zhai [7] established the corresponding results to the compressible Navier-Stokes equations (1.1) with density-dependent viscosity.

Now we turn back to the case of the basic wave patterns with vacuum states. As pointed out by Liu-Smoller [13], among the two nonlinear waves, i.e., shock and rarefaction waves, to the one-dimensional compressible isentropic Euler equations (1.2), only the rarefaction wave can be connected to vacuum. However, to our knowledge, so far there is no any results on the zero dissipation limit of the system (1.1) in the case when the Euler system (1.2) contain the rarefaction wave connected to the vacuum. In this paper, we investigate this fundamental problem and want to obtain the decay rate with respect to the viscosity ϵ . Remark that Perepelitsa [16] consider the time-asymptotic stability of solutions of 1-d compressible Navier-Stokes equations (1.1) toward rarefaction waves connected to vacuum in Lagrangian coordinate and Jiu-Wang-Xin [12] study the large time asymptotic behavior toward rarefaction waves for solutions to the 1-dimensional compressible Navier-Stokes equations (1.1) with density-dependent viscosity for general initial data whose far fields are connected by a rarefaction wave to the corresponding Euler equations with one end state being vacuum.

Now we give a description of the rarefaction wave connected to the vacuum to the compressible Euler equations (1.2), see also the references [13] and [17]. For definiteness, 2-rarefaction wave will be considered. If we investigate the compressible Euler system (1.2) with the Riemann initial data

$$\begin{cases} \rho(0, x) = 0, & x < 0, \\ (\rho, u)(0, x) = (\rho_+, u_+), & x > 0, \end{cases} \quad (1.3)$$

where the left side is the vacuum state and $\rho_+ > 0, u_+$ are prescribed constants on the right state, then the Riemann problem (1.2), (1.3) admits a 2-rarefaction wave connected to the vacuum on the left side. By the fact that along the 2-rarefaction wave curve, 2-Riemann invariant $\Sigma_2(\rho, u)$ is constant in (x, t) , we can get the velocity $u_- = \Sigma_2(\rho_+, u_+)$ being the speed of the fluid coming into the vacuum from the 2-rarefaction wave. This 2-rarefaction wave connecting the vacuum $\rho = 0$ to (ρ_+, u_+) is the self-similar solution $(\rho^{r_2}, u^{r_2})(\xi)$, $(\xi = \frac{x}{t})$ of (1.2) defined by

$$\lambda_2(\rho^{r_2}(\xi), u^{r_2}(\xi)) = \begin{cases} \rho^{r_2}(\xi) = 0, & \text{if } \xi < \lambda_2(0, u_-) = u_-, \\ \xi, & \text{if } u_- \leq \xi \leq \lambda_2(\rho_+, u_+), \\ \lambda_2(\rho_+, u_+), & \text{if } \xi > \lambda_2(\rho_+, u_+), \end{cases} \quad (1.4)$$

and

$$\Sigma_2(\rho^{r_2}(\xi), u^{r_2}(\xi)) = \Sigma_2(0, u_-) = \Sigma_2(\rho_+, u_+). \quad (1.5)$$

Thus we can define the momentum of 2-rarefaction wave by

$$m^{r_2}(\xi) = \begin{cases} \rho^{r_2}(\xi)u^{r_2}(\xi), & \text{if } \rho^{r_2} > 0, \\ 0, & \text{if } \rho^{r_2} = 0. \end{cases} \quad (1.6)$$

In the present paper, we want to construct a sequence of solutions $(\rho^\epsilon, m^\epsilon)(x, t)$ to the compressible Navier-Stokes equations (1.1) which converge to the 2-rarefaction wave $(\rho_2^r, m_2^r)(x/t)$ defined above as ϵ tends to zero. The effects of initial layers will be ignored by choosing the well-prepared initial data depending on the viscosity for the Navier-Stokes equations.

The main novelty and difficulty of the paper is how to control the degeneracies caused by the vacuum in the rarefaction wave. To overcome this difficulty, we first cut off the 2-rarefaction wave with vacuum along the rarefaction wave curve. More precisely, for any $\mu > 0$ to be determined, the cut-off rarefaction wave will connect the state $(\rho, u) = (\mu, u_\mu)$ and (ρ_+, u_+) where u_μ can be obtained uniquely by the definition of the 2-rarefaction wave curve. Then an approximate rarefaction wave to this cut-off rarefaction wave will be constructed through the Burgers equation. Finally, the desired solution sequences to the compressible Navier-Stokes equations (1.1) could be established around the approximate rarefaction wave. The uniform estimates to the perturbation of the solution sequences around the approximate rarefaction wave can be got by the following two observations. One is the fact that the viscosity ϵ can control the degeneracies caused by the vacuum in rarefaction waves by choosing suitably $\mu = \mu(\epsilon)$. In fact, we choose $\mu = \epsilon^a |\ln \epsilon|$ with a defined in (3.11) in the present paper. The other observation is that we can carry out the energy estimates under the a priori assumption that the perturbation is suitably small in $H^1(\mathbf{R})$ norm with some decay rate with respect to ϵ as ϵ tends to zero. See (3.10) in the below for the details. Note that this a priori assumption is natural but is first used in studying zero dissipation limit to our knowledge. With these two observations, we can close the a priori assumption and obtain the desired results.

Now our main result is stated as follows.

Theorem 1.1. *Let $(\rho^{r_2}, m^{r_2})(x/t)$ be the 2-rarefaction wave defined by (1.4)-(1.6) with one-side vacuum state. Then there exists a small positive constant ϵ_0 such that for any $\epsilon \in (0, \epsilon_0)$, we can construct a global smooth solution $(\rho^\epsilon, m^\epsilon = \rho^\epsilon u^\epsilon)(x, t)$ with initial values (3.1) to the compressible Navier-Stokes equation (1.1) satisfying*

$$\begin{aligned} (\rho^\epsilon - \rho^{r_2}, m^\epsilon - m^{r_2}), (\rho^\epsilon, m^\epsilon)_x &\in C^0((0, +\infty); L^2(\mathbf{R})), \\ m_{xx}^\epsilon &\in L^2(0, +\infty; L^2(\mathbf{R})). \end{aligned}$$

2) As viscosity $\epsilon \rightarrow 0$, $(\rho^\epsilon, m^\epsilon)(x, t)$ converges to $(\rho^{r_2}, m^{r_2})(x/t)$ pointwisely except the original point $(0, 0)$. Furthermore, for any given positive constant h , there exists a constant $C_h > 0$, independent of ϵ , such that

$$\begin{aligned} \sup_{t \geq h} \|\rho^\epsilon(\cdot, t) - \rho^{r_2}(\frac{\cdot}{t})\|_{L^\infty} &\leq C_h \epsilon^a |\ln \epsilon|, \\ \sup_{t \geq h} \|m^\epsilon(\cdot, t) - m^{r_2}(\frac{\cdot}{t})\|_{L^\infty} &\leq \begin{cases} C_h \epsilon^b |\ln \epsilon|^{-\frac{1}{2}}, & \text{if } 1 < \gamma < 3, \\ C_h \epsilon^{\frac{1}{\gamma+4}} |\ln \epsilon|, & \text{if } \gamma \geq 3, \end{cases} \end{aligned} \quad (1.7)$$

with the positive constants a, b given by

$$a = \begin{cases} \frac{1}{6} & \text{if } 1 < \gamma \leq 2, \\ \frac{1}{\gamma+4}, & \text{if } \gamma > 2. \end{cases} \quad (1.8)$$

and

$$b = \begin{cases} \frac{1}{8} & \text{if } 1 < \gamma \leq 2, \\ \frac{\gamma+1}{4(\gamma+4)}, & \text{if } 2 < \gamma < 3. \end{cases} \quad (1.9)$$

A few remarks are followed.

Remark 1.2. Similar result to Theorem 1.1 is also expected for a one-dimensional compressible Navier-Stokes equation with density-dependent viscosity which reads

$$\begin{cases} \rho_t + (\rho u)_x = 0, \\ (\rho u)_t + (\rho u^2 + \rho^\gamma)_x = \epsilon (\rho^\alpha u_x)_x, \end{cases} \quad (1.10)$$

with suitable $\alpha > 0$ and $\gamma > 1$. Actually, the system (1.10) can be derived by Chapman-Enskog expansions from the Boltzmann equation where the viscosity of the compressible Navier-Stokes equations depends on the temperature and thus on the density for isentropic flows. Also, the viscous Saint-Venant system for the shallow water, derived from the incompressible Navier-Stokes equation with a moving free surface, is expressed exactly as in (1.10) with $\alpha = 1$ and $\gamma = 2$, see [2, 4]. In this situation, since viscosity vanishes at vacuum, the convergence rate with respect to ϵ may become slower than in Theorem 1.1 and may depend on α and γ . However, this is left to the forthcoming paper.

Remark 1.3. Our result and method can also be generalized to the 1-D full compressible Navier-Stokes equations with the conservation of the energy and the Boltzmann equation with slab symmetry. This is left to the forthcoming paper.

Remark 1.4. *It is also interesting to study the zero dissipation limit of compressible Navier-Stokes equations (1.1) in the case when the Euler system (1.2) has two rarefaction waves with the vacuum states in the middle. However, it is nontrivial to cut off these rarefaction waves with vacuum along the corresponding rarefaction wave curves. In fact, the wave structure containing two rarefaction waves with the medium vacuum is destroyed and some new wave may occur in the cut-off process, which is quite different from the single rarefaction wave case considered in the present paper.*

Remark 1.5. *It is noted that in the a priori estimates (3.17) below, the estimates for ϕ^2 from the potential energy holds with the weight $\bar{\rho}^{\gamma-2}$ which is degenerate at vacuum when $\gamma > 2$. Therefore, the convergence rate obtained in Lemma 3.2 and thus in Theorem 1.1 depends on γ when $\gamma > 2$.*

The rest of the paper is organized as follows. In section 2, we construct a smooth 2-rarefaction wave which approximates the cut-off rarefaction wave based on the inviscid Burgers equation. And the proof the Theorem 1.1 is given in Section 3.

Throughout this paper, $H^l(\mathbf{R})$, $l = 0, 1, 2, \dots$, denotes the l -th order Sobolev space with its norm

$$\|f\|_l = \left(\sum_{j=0}^l \|\partial_y^j f\|^2 \right)^{\frac{1}{2}}, \quad \text{and } \|\cdot\| := \|\cdot\|_{L^2(dy)},$$

while $L^2(dz)$ means the L^2 integral over \mathbf{R} with respect to the Lebesgue measure dz , and $z = x$ or y . For simplicity, we also write C as generic positive constants which are independent of time t and viscosity ϵ unless otherwise stated.

2 Rarefaction waves

Since there is no exact rarefaction wave profile for the Navier-Stokes equations (1.1), the following approximate rarefaction wave profile satisfying the Euler equations was motivated by Matsumura-Nishihara [15] and Xin [20]. For the completeness of the presentation, we include its definition and the properties listed in Lemma 2.1. However, Lemma 2.1 is a little different from [20] as stated after Lemma 2.1.

Consider the Riemann problem for the inviscid Burgers equation:

$$\begin{cases} w_t + ww_x = 0, \\ w(x, 0) = \begin{cases} w_-, & x < 0, \\ w_+, & x > 0. \end{cases} \end{cases} \quad (2.1)$$

If $w_- < w_+$, then the Riemann problem (2.1) admits a rarefaction wave solution $w^r(x, t) = w^r(\frac{x}{t})$ given by

$$w^r\left(\frac{x}{t}\right) = \begin{cases} w_-, & \frac{x}{t} \leq w_-, \\ \frac{x}{t}, & w_- \leq \frac{x}{t} \leq w_+, \\ w_+, & \frac{x}{t} \geq w_+. \end{cases} \quad (2.2)$$

As in [20], the approximate rarefaction wave to the compressible Navier-Stokes equations (1.1) can be constructed by the solution of the Burgers equation

$$\begin{cases} w_t + ww_x = 0, \\ w(0, x) = w_\delta(x) = w\left(\frac{x}{\delta}\right) = \frac{w_+ + w_-}{2} + \frac{w_+ - w_-}{2} \tanh \frac{x}{\delta}, \end{cases} \quad (2.3)$$

where $\delta > 0$ is a small parameter to be determined. In fact, we choose $\delta = \epsilon^a$ in (3.12) with a given by (3.11) in the following. Note that the solution $w_\delta^r(t, x)$ of the problem (2.3) is given by

$$w_\delta^r(t, x) = w_\delta(x_0(t, x)), \quad x = x_0(t, x) + w_\delta(x_0(t, x))t. \quad (2.4)$$

And $w_\delta^r(t, x)$ has the following properties:

Lemma 2.1. *The problem (2.3) has a unique smooth global solution $w_\delta^r(x, t)$ for each $\delta > 0$ such that*

(1) $w_- < w_\delta^r(x, t) < w_+$, $\partial_x w_\delta^r(x, t) > 0$, for $x \in \mathbf{R}$, $t \geq 0$, $\delta > 0$.

(2) The following estimates hold for all $t > 0$, $\delta > 0$ and $p \in [1, \infty]$:

$$\|\partial_x w_\delta^r(\cdot, t)\|_{L^p} \leq C(w_+ - w_-)^{1/p} (\delta + t)^{-1+1/p}, \quad (2.5)$$

$$\|\partial_x^2 w_\delta^r(\cdot, t)\|_{L^p} \leq C(\delta + t)^{-1} \delta^{-1+1/p}, \quad (2.6)$$

$$\left| \frac{\partial^2 w_\delta^r(x, t)}{\partial x^2} \right| \leq \frac{4}{\delta} \frac{\partial w_\delta^r(x, t)}{\partial x}. \quad (2.7)$$

(3) There exist a constant $\delta_0 \in (0, 1)$ such that for $\delta \in (0, \delta_0]$, $t > 0$,

$$\|w_\delta^r(\cdot, t) - w^r(\frac{\cdot}{t})\|_{L^\infty} \leq C\delta t^{-1} [\ln(1+t) + |\ln \delta|].$$

The proof of Lemma 2.1 can be found in Xin [20]. However, the description of Lemma 2.1 is equivalent to but a little different from Xin [20]. Take the estimation (2.5) as an example, which is described by

$$\|\partial_x w_\delta^r(\cdot, t)\|_{L^p} \leq C \min\{(w_+ - w_-)\delta^{-1+1/p}, (w_+ - w_-)^{1/p} t^{-1+1/p}\}, \quad (2.5)'$$

in Xin [20]. In fact, two estimations (2.5) and (2.5)' are equivalent for fixed wave strength $w_+ - w_-$. However, the advantage of Lemma 2.1 is that the energy estimate can be carried out for all time since there is no singularity to the approximate rarefaction wave even at $t = 0$. While in Xin's paper [20], the energy estimate must be done in two time-scalings, that is, finite time and large time, due to the singularity of the estimations of the approximate rarefaction wave at $t = 0$.

As mentioned in the introduction, we will cut off the 2-rarefaction wave with vacuum along the wave curve in order to overcome the difficulty caused by the vacuum,. More precisely, for any $\mu > 0$ to be determined, we can get a state $(\rho, u) = (\mu, u_\mu)$ belonging to the 2-rarefaction wave curve. From the fact that 2-Riemann invariant $\Sigma_2(\rho, u)$ is constant along the 2-rarefaction wave curve, u_μ can be computed explicitly by $u_\mu = \Sigma_2(\rho_+, u_+) + \frac{2}{\gamma-1}\mu^{\frac{\gamma-1}{2}}$. Now we get a new 2-rarefaction wave $(\rho_\mu^{r2}, u_\mu^{r2})(\xi)$, ($\xi = x/t$) connecting the state (μ, u_μ) to the state (ρ_+, u_+) which can be expressed explicitly by

$$\lambda_2(\rho_\mu^{r2}, u_\mu^{r2})(\xi) = \begin{cases} \lambda_2(\mu, u_\mu), & \xi < \lambda_2(\mu, u_\mu), \\ \xi, & \lambda_2(\mu, u_\mu) \leq \xi \leq \lambda_2(\rho_+, u_+), \\ \lambda_2(\rho_+, u_+), & \xi > \lambda_2(\rho_+, u_+). \end{cases} \quad (2.8)$$

and

$$\Sigma_2(\rho_\mu^{r2}, u_\mu^{r2}) = \Sigma_2(\mu, u_\mu) = \Sigma_2(\rho_+, u_+). \quad (2.9)$$

Correspondingly, we can define the momentum function $m_\mu^{r2} = \rho_\mu^{r2} u_\mu^{r2}$. It is easy to show that the cut-off 2-rarefaction wave $(\rho_\mu^{r2}, m_\mu^{r2})(x/t)$ converges to the original 2-rarefaction wave with vacuum $(\rho^{r2}, m^{r2})(x/t)$ in sup-norm with the convergence rate μ as μ tends to zero. More precisely, we have

Lemma 2.2. *There exist a constant $\mu_0 \in (0, 1)$ such that for $\mu \in (0, \mu_0]$, $t > 0$,*

$$\|(\rho_\mu^{r2}, m_\mu^{r2})(\cdot/t) - (\rho^{r2}, m^{r2})(\cdot/t)\|_{L^\infty} \leq C\mu.$$

The proof of Lemma 2.2 can be obtained directly from the explicit solution formula of rarefaction waves, so we omit it for brevity.

Now the approximate rarefaction wave $(\bar{\rho}_{\mu,\delta}, \bar{u}_{\mu,\delta})(x, t)$ of the cut-off 2-rarefaction wave $(\rho_\mu^{r_2}, u_\mu^{r_2})(\frac{x}{t})$ to compressible Navier-Stokes equations (1.1) can be defined by

$$\begin{cases} w_+ = \lambda_2(\rho_+, u_+), & w_- = \lambda_2(\mu, u_\mu), \\ w_\delta^r(t, x) = \lambda_2(\bar{\rho}_{\mu,\delta}, \bar{u}_{\mu,\delta})(t, x), \\ \Sigma_2(\bar{\rho}_{\mu,\delta}, \bar{u}_{\mu,\delta})(x, t) = \Sigma_2(\rho_+, u_+) = \Sigma_2(\mu, u_\mu), \end{cases} \quad (2.10)$$

where w_δ^r is the solution of Burger's equation (2.3) defined in (2.4). From then on, the subscription of $(\bar{\rho}_{\delta,\mu}, \bar{u}_{\delta,\mu})(x, t)$ will be omitted as $(\bar{\rho}, \bar{u})(x, t)$ for simplicity. Then the approximate cut-off 2-rarefaction wave $(\bar{\rho}, \bar{u})$ defined above satisfies

$$\begin{cases} \bar{\rho}_t + (\bar{\rho}\bar{u})_x = 0, \\ (\bar{\rho}\bar{u})_t + (\bar{\rho}\bar{u}^2 + p(\bar{\rho}))_x = 0, \end{cases} \quad (2.11)$$

and the properties of the approximate rarefaction wave $(\bar{\rho}, \bar{u})$ is listed without proof in the following Lemma.

Lemma 2.3. *The approximate cut-off 2-rarefaction wave $(\bar{\rho}, \bar{u})$ defined in (2.10) satisfies the following properties:*

$$(i) \quad \bar{u}_x(x, t) = \frac{2}{\gamma+1}(w_\delta^r)_x > 0, \text{ for } x \in \mathbf{R}, t \geq 0;$$

$$\bar{\rho}_x = \bar{\rho}^{\frac{3-\gamma}{2}}\bar{u}_x, \text{ and } \bar{\rho}_{xx} = \bar{\rho}^{\frac{3-\gamma}{2}}\bar{u}_{xx} + \frac{3-\gamma}{2}\bar{\rho}^{2-\gamma}(\bar{u}_x)^2.$$

(ii) *The following estimates hold for all $t > 0$, $\delta > 0$ and $p \in [1, \infty]$:*

$$\|\bar{u}_x(\cdot, t)\|_{L^p} \leq C(w_+ - w_-)^{1/p}(\delta + t)^{-1+1/p},$$

$$\|\bar{u}_{xx}(\cdot, t)\|_{L^p} \leq C(\delta + t)^{-1}\delta^{-1+1/p}.$$

(iii) *There exist a constant $\delta_0 \in (0, 1)$ such that for $\delta \in (0, \delta_0]$, $t > 0$,*

$$\|(\bar{\rho} - \rho_\mu^{r_2}, \bar{u} - u_\mu^{r_2})(\cdot, t)\|_{L^\infty} \leq C\delta t^{-1}[\ln(1+t) + |\ln \delta|].$$

3 Proof of Theorem 1.1

To prove Theorem 1.1, the solution $(\rho^\epsilon, u^\epsilon)$ is constructed as the perturbation around the approximate rarefaction wave $(\bar{\rho}, \bar{u})$. Consider the Cauchy problem for (1.1) with

smooth initial data

$$(\rho^\epsilon, u^\epsilon)(x, t = 0) = (\bar{\rho}, \bar{u})(x, 0). \quad (3.1)$$

Then we introduce the perturbation

$$(\phi, \psi)(y, \tau) = (\rho^\epsilon, u^\epsilon)(x, t) - (\bar{\rho}, \bar{u})(x, t), \quad (3.2)$$

where y, τ are the scaled variables as

$$y = \frac{x}{\epsilon}, \quad \tau = \frac{t}{\epsilon}, \quad (3.3)$$

and $(\rho^\epsilon, u^\epsilon)$ is assumed to be the solution to the problem (1.1). For the simplicity of the notation, the superscription of $(\rho^\epsilon, u^\epsilon)$ will be omitted as (ρ, u) from now on if there is no confusion of the notation. Substituting (3.2) and (3.3) into (1.1) and using the definition for $(\bar{\rho}, \bar{u})$, we obtain

$$\phi_\tau + \rho\psi_y + u\phi_y = -f, \quad (3.4)$$

$$\rho\psi_\tau + \rho u\psi_y + p'(\rho)\phi_y - \psi_{yy} = -g, \quad (3.5)$$

$$(\phi, \psi)(y, 0) = 0, \quad (3.6)$$

where

$$\begin{cases} f = \bar{u}_y\phi + \bar{\rho}_y\psi, \\ g = -\bar{u}_{yy} + \rho\psi\bar{u}_y + \bar{\rho}_y\left[p'(\rho) - \frac{\rho}{\bar{\rho}}p'(\bar{\rho})\right]. \end{cases} \quad (3.7)$$

We seek a global (in time) solution (ϕ, ψ) to the reformulated problem (3.4) – (3.6). To this end, the solution space for (3.4) – (3.6) is defined by

$$X(0, \tau_1) = \left\{ (\phi, \psi) \middle| (\phi, \psi) \in C^0([0, \tau_1]; H^1(\mathbf{R})), \quad \phi_y \in L^2(0, \tau_1; L^2(\mathbf{R})), \right. \\ \left. \psi_y \in L^2(0, \tau_1; H^1(\mathbf{R})) \right\}$$

with $0 < \tau_1 \leq +\infty$.

Theorem 3.1. *The problem (3.4) – (3.6) admits a unique global-in-time solution $(\phi, \psi) \in X(0, +\infty)$. Furthermore, there exist positive constants ϵ_0 and C independent of ϵ , such that if $0 < \epsilon \leq \epsilon_0$, then*

$$\begin{aligned} & \sup_{\tau \in [0, +\infty]} \int_{\mathbf{R}} \left(\bar{\rho}\psi^2 + \bar{\rho}^{\gamma-2}\phi^2 + \phi_y^2 + \psi_y^2 \right)(\tau, y) dy \\ & + \int_0^{+\infty} \int_{\mathbf{R}} \left[\psi_y^2 + \bar{\rho}^{\gamma-2}\bar{u}_y\phi^2 + \bar{\rho}\bar{u}_y\psi^2 + \bar{\rho}^{\gamma-3}\phi_y^2 + \frac{\psi_{yy}^2}{\bar{\rho}} \right] dy d\tau \leq C\epsilon^{(1/2-a)} |\ln \epsilon|^{-1/2}. \end{aligned} \quad (3.8)$$

Consequently,

$$\begin{aligned} \sup_{0 \leq \tau \leq +\infty} \|\phi(\cdot, \tau)\|_{L^\infty} &\leq \begin{cases} C\epsilon^{1/6} |\ln \epsilon|^{-1/4}, & \text{if } 1 < \gamma \leq 2, \\ C\epsilon^{\frac{1}{\gamma+4}} |\ln \epsilon|^{(1-\gamma)/4}, & \text{if } \gamma > 2, \end{cases} \\ \sup_{0 \leq \tau \leq +\infty} \|\psi(\cdot, \tau)\|_{L^\infty} &\leq \begin{cases} C\epsilon^{1/8} |\ln \epsilon|^{-1/2}, & \text{if } 1 < \gamma \leq 2, \\ C\epsilon^{\frac{\gamma+1}{4(\gamma+4)}} |\ln \epsilon|^{-1/2}, & \text{if } \gamma > 2. \end{cases} \end{aligned} \quad (3.9)$$

In what follows, the analysis is always carried out under the a priori assumptions

$$\sup_{0 \leq \tau \leq \tau_1} \|\phi(\cdot, \tau)\|_{L^\infty} \leq \epsilon^a, \quad \sup_{\tau \in [0, \tau_1]} \|\psi_y\| \leq 1, \quad (3.10)$$

with a given by

$$a = \begin{cases} \frac{1}{6}, & 1 < \gamma \leq 2, \\ \frac{1}{\gamma+4}, & \gamma > 2. \end{cases} \quad (3.11)$$

Take

$$\mu = \epsilon^a |\ln \epsilon|, \quad \delta = \epsilon^a, \quad (3.12)$$

in the sequel. Then it follows that $\mu \geq 2\epsilon^a$ if $\epsilon \ll 1$. Under the a priori assumption (3.10), we can get

$$\frac{\bar{\rho}}{2} \leq \rho \leq \frac{3\bar{\rho}}{2}. \quad (3.13)$$

In fact, if $\epsilon \ll 1$, then

$$\rho = \bar{\rho} + \phi \geq \bar{\rho} - \|\phi\|_{L^\infty} \geq \bar{\rho} - \epsilon^a \geq \bar{\rho} - \frac{1}{2}\mu \geq \frac{\bar{\rho}}{2}, \quad (3.14)$$

$$\rho = \bar{\rho} + \phi \leq \bar{\rho} + \|\phi\|_{L^\infty} \leq \bar{\rho} + \epsilon^a \leq \bar{\rho} + \frac{1}{2}\mu \leq \frac{3\bar{\rho}}{2}. \quad (3.15)$$

Moreover, under the a priori assumption (3.10), it holds that

$$C_1 \bar{\rho}^{\gamma-2} \phi^2 \leq p(\rho) - p(\bar{\rho}) - p'(\bar{\rho})\phi \leq C_2 \bar{\rho}^{\gamma-2} \phi^2. \quad (3.16)$$

where C_1, C_2 are positive constants independent of ϵ .

Since the proof for the local existence of the solution to (3.4) – (3.6) is standard, we omit it for brevity. To prove Theorem 3.1, it is sufficient to prove the following a priori estimates.

Lemma 3.2. (*A priori estimates*) Let $\gamma > 1$ and $(\phi, \psi) \in X(0, \tau_1)$ be a solution to the problem (3.4) – (3.6). Then under the a priori assumption (3.10), there exist positive constants ϵ_0 and C independent of ϵ , such that if $0 < \epsilon \leq \epsilon_0$, then

$$\begin{aligned} & \sup_{\tau \in [0, \tau_1]} \int_{\mathbf{R}} \left(\bar{\rho} \psi^2 + \bar{\rho}^{\gamma-2} \phi^2 + \phi_y^2 + \psi_y^2 \right) (\tau, y) dy \\ & + \int_0^{\tau_1} \int_{\mathbf{R}} \left[\psi_y^2 + \bar{\rho}^{\gamma-2} \bar{u}_y \phi^2 + \bar{\rho} \bar{u}_y \psi^2 + \bar{\rho}^{\gamma-3} \phi_y^2 + \frac{\psi_{yy}^2}{\bar{\rho}} \right] dy d\tau \leq C \epsilon^{(1/2-a)} |\ln \epsilon|^{-1/2}. \end{aligned} \quad (3.17)$$

Consequently,

$$\begin{aligned} \sup_{0 \leq \tau \leq \tau_1} \|\phi(\cdot, \tau)\|_{L^\infty} & \leq \begin{cases} C \epsilon^{1/6} |\ln \epsilon|^{-1/4}, & \text{if } 1 < \gamma \leq 2, \\ C \epsilon^{\frac{1}{\gamma+4}} |\ln \epsilon|^{(1-\gamma)/4}, & \text{if } \gamma > 2, \end{cases} \\ \sup_{0 \leq \tau \leq \tau_1} \|\psi(\cdot, \tau)\|_{L^\infty} & \leq \begin{cases} C \epsilon^{1/8} |\ln \epsilon|^{-1/2}, & \text{if } 1 < \gamma \leq 2, \\ C \epsilon^{\frac{\gamma+1}{4(\gamma+4)}} |\ln \epsilon|^{-1/2}, & \text{if } \gamma > 2. \end{cases} \end{aligned} \quad (3.18)$$

Proof of Lemma 3.2: The proof of Lemma 3.2 consists of the following steps.

Step 1. First, define

$$E := \Phi(\rho, \bar{\rho}) + \frac{\psi^2}{2},$$

where

$$\Phi(\rho, \bar{\rho}) := \int_{\bar{\rho}}^{\rho} \frac{p(\xi) - p(\bar{\rho})}{\xi^2} d\xi = \frac{1}{(\gamma-1)\rho} (p(\rho) - p(\bar{\rho}) - p'(\bar{\rho})\phi). \quad (3.19)$$

Direct computations yield

$$\begin{aligned} & \left(\rho E \right)_\tau + \left[\rho u E - \psi_y \psi + (p(\rho) - p(\bar{\rho})) \psi \right]_y \\ & + \psi_y^2 + \bar{u}_y (p(\rho) - p(\bar{\rho}) - p'(\bar{\rho})\phi) + \psi^2 \rho \bar{u}_y = \bar{u}_{yy} \psi. \end{aligned}$$

Then integrating the above equation over $\mathbf{R}^1 \times [0, \tau]$ and using (3.13), (3.16) and (3.19) imply

$$\int_{\mathbf{R}} \left(\bar{\rho} \psi^2 + \bar{\rho}^{\gamma-2} \phi^2 \right) dy + \int_0^\tau \int_{\mathbf{R}} \left(\psi_y^2 + \bar{\rho}^{\gamma-2} \bar{u}_y \phi^2 + \bar{\rho} \bar{u}_y \psi^2 \right) dy d\tau \leq C \int_0^\tau \int_{\mathbf{R}} |\bar{u}_{yy} \psi| dy d\tau. \quad (3.20)$$

By Sobolev inequality and Lemma 2.3, one has

$$\begin{aligned}
& \int_0^\tau \int_{\mathbf{R}} |\bar{u}_{yy}\psi| dy d\tau \leq C \int_0^\tau \|\bar{u}_{yy}\|_{L^1} \|\psi\|^{1/2} \|\psi_y\|^{1/2} d\tau \\
& \leq C \int_0^\tau \frac{1}{\tau + \delta/\epsilon} \|\psi\|^{1/2} \|\psi_y\|^{1/2} d\tau \\
& \leq \frac{1}{8} \int_0^\tau \|\psi_y\|^2 d\tau + C \int_0^\tau \left(\frac{1}{\tau + \delta/\epsilon}\right)^{4/3} \|\psi\|^{2/3} d\tau \\
& \leq \frac{1}{8} \int_0^\tau \|\psi_y\|^2 d\tau + \frac{1}{8} \sup_{\tau \in [0, \tau_1]} \|\sqrt{\bar{\rho}}\psi\|^2 + C \left(\mu^{-1/3} \int_0^\infty \left(\frac{1}{\tau + \delta/\epsilon}\right)^{4/3} d\tau\right)^{3/2} \\
& \leq \frac{1}{8} \int_0^\tau \|\psi_y\|^2 d\tau + \frac{1}{8} \sup_{\tau \in [0, \tau_1]} \|\sqrt{\bar{\rho}}\psi\|^2 + C \left(\frac{\epsilon}{\mu\delta}\right)^{1/2}.
\end{aligned} \tag{3.21}$$

Combining (3.20) and (3.21) and recalling (3.12) yield

$$\begin{aligned}
& \sup_{\tau \in [0, \tau_1]} \int_{\mathbf{R}} \left(\bar{\rho}\psi^2 + \bar{\rho}^{\gamma-2}\phi^2 \right) (\tau, y) dy + \int_0^{\tau_1} \int_{\mathbf{R}} \left[\psi_y^2 + \bar{\rho}^{\gamma-2}\bar{u}_y\phi^2 + \bar{\rho}\bar{u}_y\psi^2 \right] dy d\tau \\
& \leq C\epsilon^{(1/2-a)} |\ln \epsilon|^{-1/2}.
\end{aligned} \tag{3.22}$$

Step 2. We make use of the idea in [8] with modifications to derive the estimation of ϕ_y . Differentiating (3.4) with respect to y and then multiplying the resulted equation by ϕ_y/ρ^3 to get

$$\left(\frac{\phi_y^2}{2\rho^3}\right)_\tau + \left(\frac{u\phi_y^2}{2\rho^3}\right)_y + \frac{\psi_{yy}\phi_y}{\rho^2} = -\frac{\phi_y}{\rho^3} (\bar{u}_{yy}\phi + \bar{\rho}_{yy}\psi + 2\bar{\rho}_y\psi_y). \tag{3.23}$$

Multiplying (3.5) by ϕ_y/ρ^2 gives

$$\begin{aligned}
& \left(\frac{\psi\phi_y}{\rho}\right)_\tau - \left(\frac{\psi\phi_\tau}{\rho} + \bar{\rho}_y\frac{\psi^2}{\rho}\right)_y - \psi_y^2 + p'(\rho)\frac{\phi_y^2}{\rho^2} - \frac{\psi_{yy}\phi_y}{\rho^2} \\
& - \bar{u}_y\frac{\psi_y\phi}{\rho} + 2\bar{\rho}_y\frac{\psi\psi_y}{\rho} + \bar{\rho}_{yy}\frac{\psi^2}{\rho} + \bar{\rho}_y\bar{u}_y\frac{\psi\phi}{\rho^2} - \bar{\rho}\bar{u}_y\frac{\psi\phi_y}{\rho^2} = -g\frac{\phi_y}{\rho^2}.
\end{aligned} \tag{3.24}$$

Adding (3.23) and (3.24) together, then integrating the resulted equation over $\mathbf{R}^1 \times [0, \tau]$ imply

$$\begin{aligned}
& \int_{\mathbf{R}} \left(\frac{\phi_y^2}{2\rho^3} + \frac{\psi\phi_y}{\rho} \right) dy + \int_0^\tau \int_{\mathbf{R}} p'(\rho) \frac{\phi_y^2}{\rho^2} dy d\tau \\
& = \int_0^\tau \int_{\mathbf{R}} \left\{ \psi_y^2 + \bar{u}_y\frac{\psi_y\phi}{\rho} - 2\bar{\rho}_y\frac{\psi\psi_y}{\rho} - \bar{\rho}_{yy}\frac{\psi^2}{\rho} - \bar{\rho}_y\bar{u}_y\frac{\psi\phi}{\rho^2} + \bar{\rho}\bar{u}_y\frac{\psi\phi_y}{\rho^2} \right. \\
& \quad \left. - \frac{\phi_y}{\rho^3} (\bar{u}_{yy}\phi + \bar{\rho}_{yy}\psi + 2\bar{\rho}_y\psi_y) - g\frac{\phi_y}{\rho^2} \right\} dy d\tau.
\end{aligned} \tag{3.25}$$

The combination of (3.22) and (3.25) leads to

$$\begin{aligned}
& \int_{\mathbf{R}} \left(\frac{\phi_y^2}{\bar{\rho}^3} + \bar{\rho}\psi^2 + \bar{\rho}^{\gamma-2}\phi^2 \right) dy + \int_0^\tau \int_{\mathbf{R}} \left(\psi_y^2 + \bar{\rho}^{\gamma-2}\bar{u}_y\phi^2 + \bar{\rho}\bar{u}_y\psi^2 + \bar{\rho}^{\gamma-3}\phi_y^2 \right) dy d\tau \\
& \leq C \int_0^\tau \int_{\mathbf{R}} \left\{ |\bar{u}_y \frac{\psi_y\phi}{\bar{\rho}}| + |\bar{\rho}_y \frac{\psi\psi_y}{\bar{\rho}}| + |\bar{\rho}_{yy} \frac{\psi^2}{\bar{\rho}}| + |\bar{\rho}_y \bar{u}_y \frac{\psi\phi}{\bar{\rho}^2}| + |\bar{u}_y \frac{\psi\phi_y}{\bar{\rho}}| + |\bar{\rho}_y \frac{\phi_y}{\bar{\rho}^3} \psi_y| \right. \\
& \quad \left. + |\frac{\bar{u}_{yy}}{\bar{\rho}^3} \phi_y\phi| + |\frac{\bar{\rho}_{yy}}{\bar{\rho}^3} \phi_y\psi| + |g \frac{\phi_y}{\bar{\rho}^2}| \right\} dy d\tau + C\epsilon^{(1/2-a)} |\ln \epsilon|^{-1/2} \\
& := \sum_{i=1}^9 I_i + C\epsilon^{(1/2-a)} |\ln \epsilon|^{-1/2}.
\end{aligned} \tag{3.26}$$

Now we estimate the terms on the right hand side of (3.26) one by one. By Lemma 2.3, (3.13) and Cauchy's inequality, it holds that

$$\begin{aligned}
I_1 &= \int_0^\tau \int_{\mathbf{R}} |\bar{u}_y \frac{\psi_y\phi}{\bar{\rho}}| dy d\tau \leq \frac{1}{8} \int_0^\tau \|\psi_y\|^2 d\tau + C \int_0^\tau \int_{\mathbf{R}} \bar{\rho}^{\gamma-2} \bar{u}_y \phi^2 \frac{\bar{u}_y}{\bar{\rho}^\gamma} dy d\tau \\
&\leq \frac{1}{8} \int_0^\tau \|\psi_y\|^2 d\tau + C\mu^{-\gamma}\delta^{-1}\epsilon \int_0^\tau \int_{\mathbf{R}} \bar{\rho}^{\gamma-2} \bar{u}_y \phi^2 dy d\tau \\
&\leq \frac{1}{8} \int_0^\tau \|\psi_y\|^2 d\tau + \frac{1}{8} \int_0^\tau \int_{\mathbf{R}} \bar{\rho}^{\gamma-2} \bar{u}_y \phi^2 dy d\tau
\end{aligned} \tag{3.27}$$

where we have used the fact that

$$C\mu^{-\gamma}\delta^{-1}\epsilon = C\epsilon^{1-a-a\gamma} |\ln \epsilon|^{-\gamma} \leq C\epsilon^{\frac{1}{2}} |\ln \epsilon|^{-\gamma} \leq \frac{1}{8}, \quad \text{if } \epsilon \ll 1. \tag{3.28}$$

From Lemma 2.3 (i), one has

$$\bar{\rho}_y = \bar{\rho}^{\frac{3-\gamma}{2}} \bar{u}_y, \tag{3.29}$$

one can get

$$\begin{aligned}
I_2 &= \int_0^\tau \int_{\mathbf{R}} |\bar{\rho}_y \frac{\psi\psi_y}{\bar{\rho}}| dy d\tau \\
&\leq \frac{1}{8} \int_0^\tau \|\psi_y\|^2 d\tau + C \int_0^\tau \int_{\mathbf{R}} \bar{\rho}^{-\gamma} \bar{u}_y (\bar{\rho} \bar{u}_y \psi^2) dy d\tau \\
&\leq \frac{1}{8} \int_0^\tau \|\psi_y\|^2 d\tau + C\mu^{-\gamma}\delta^{-1}\epsilon \int_0^\tau \int_{\mathbf{R}} \bar{\rho} \bar{u}_y \psi^2 dy d\tau \\
&\leq \frac{1}{8} \int_0^\tau \|\psi_y\|^2 d\tau + \frac{1}{8} \int_0^\tau \int_{\mathbf{R}} \bar{\rho} \bar{u}_y \psi^2 dy d\tau
\end{aligned} \tag{3.30}$$

where in the last inequality we used the fact (3.28).

Recalling (2.7) from Lemma 2.1 and the fact (i) in Lemma 2.3, one can arrive at

$$\begin{aligned}
|\bar{\rho}_{xx}| &= \left| \frac{2}{\gamma+1} \bar{\rho}^{\frac{3-\gamma}{2}} \omega_{\delta xx}^r + \frac{2(3-\gamma)}{(\gamma+1)^2} \bar{\rho}^{2-\gamma} (\omega_{\delta x}^r)^2 \right| \\
&\leq C \left(\bar{\rho}^{\frac{3-\gamma}{2}} \frac{\bar{u}_x}{\delta} + \bar{\rho}^{2-\gamma} \bar{u}_x^2 \right).
\end{aligned} \tag{3.31}$$

Thus one has

$$\begin{aligned}
I_3 &= \int_0^\tau \int_{\mathbf{R}} |\bar{\rho}_{yy} \frac{\psi^2}{\bar{\rho}}| dy d\tau \\
&\leq C\epsilon\delta^{-1} \int_0^\tau \int_{\mathbf{R}} |\bar{\rho}\bar{u}_y\psi^2 \bar{\rho}^{-\frac{\gamma+1}{2}}| dy d\tau + C\epsilon\delta^{-1} \int_0^\tau \int_{\mathbf{R}} |\bar{\rho}\bar{u}_y\psi^2 \bar{\rho}^{-\gamma}| dy d\tau \\
&\leq C\mu^{-\gamma}\delta^{-1}\epsilon \int_0^\tau \int_{\mathbf{R}} \bar{\rho}\bar{u}_y\psi^2 dy d\tau \\
&\leq \frac{1}{8} \int_0^\tau \int_{\mathbf{R}} \bar{\rho}\bar{u}_y\psi^2 dy d\tau, \quad \text{if } \epsilon \ll 1.
\end{aligned} \tag{3.32}$$

It follows from Lemma 2.3 and (3.29) that

$$\begin{aligned}
I_4 &= \int_0^\tau \int_{\mathbf{R}} |\bar{\rho}_y \bar{u}_y \frac{\psi\phi}{\bar{\rho}^2}| dy d\tau \\
&\leq \frac{1}{8} \int_0^\tau \int_{\mathbf{R}} \bar{\rho}\bar{u}_y\psi^2 dy d\tau + C \int_0^\tau \int_{\mathbf{R}} \bar{u}_y \bar{\rho}^{\gamma-2} \phi^2 \frac{\bar{\rho}_y^2}{\bar{\rho}^{3+\gamma}} dy d\tau \\
&\leq \frac{1}{8} \int_0^\tau \int_{\mathbf{R}} \bar{\rho}\bar{u}_y\psi^2 dy d\tau + C \frac{\epsilon^2}{\mu^{2\gamma}\delta^2} \int_0^\tau \int_{\mathbf{R}} \bar{\rho}^{\gamma-2} \bar{u}_y \phi^2 dy d\tau \\
&\leq \frac{1}{8} \int_0^\tau \int_{\mathbf{R}} \bar{\rho}\bar{u}_y\psi^2 dy d\tau + \frac{1}{8} \int_0^\tau \int_{\mathbf{R}} \bar{\rho}^{\gamma-2} \bar{u}_y \phi^2 dy d\tau.
\end{aligned} \tag{3.33}$$

Similarly, it holds that

$$\begin{aligned}
I_5 &= \int_0^\tau \int_{\mathbf{R}} |\bar{u}_y \frac{\psi\phi_y}{\bar{\rho}}| dy d\tau \\
&\leq \frac{1}{8} \int_0^\tau \|\bar{\rho}^{\frac{\gamma-3}{2}} \phi_y\|^2 d\tau + C \int_0^\tau \int_{\mathbf{R}} \bar{\rho}\bar{u}_y\psi^2 \frac{\bar{u}_y}{\bar{\rho}^\gamma} dy d\tau \\
&\leq \frac{1}{8} \int_0^\tau \|\bar{\rho}^{\frac{\gamma-3}{2}} \phi_y\|^2 d\tau + C \frac{\epsilon}{\delta\mu^\gamma} \int_0^\tau \int_{\mathbf{R}} \bar{\rho}\bar{u}_y\psi^2 dy d\tau \\
&\leq \frac{1}{8} \int_0^\tau \|\bar{\rho}^{\frac{\gamma-3}{2}} \phi_y\|^2 d\tau + \frac{1}{8} \int_0^\tau \int_{\mathbf{R}} \bar{\rho}\bar{u}_y\psi^2 dy d\tau.
\end{aligned} \tag{3.34}$$

By Lemma 2.3, the equality (3.29) and Cauchy's inequality, one has

$$\begin{aligned}
I_6 &= \int_0^\tau \int_{\mathbf{R}} |\bar{\rho}_y \frac{\phi_y}{\bar{\rho}^3} \psi_y| dy d\tau \\
&\leq \frac{1}{8} \int_0^\tau \|\bar{\rho}^{\frac{\gamma-3}{2}} \phi_y\|^2 d\tau + C \int_0^\tau \int_{\mathbf{R}} \frac{\bar{u}_y^2}{\bar{\rho}^{2\gamma}} \psi_y^2 dy d\tau \\
&\leq \frac{1}{8} \int_0^\tau \|\rho^{(\gamma-3)/2} \phi_y\|^2 d\tau + C \frac{\epsilon^2}{\delta^2 \mu^{2\gamma}} \int_0^\tau \int_{\mathbf{R}} \psi_y^2 dy d\tau \\
&\leq \frac{1}{8} \int_0^\tau \|\rho^{(\gamma-3)/2} \phi_y\|^2 d\tau + \frac{1}{8} \int_0^\tau \int_{\mathbf{R}} \psi_y^2 dy d\tau.
\end{aligned} \tag{3.35}$$

Similarly, I_7 can be estimated as

$$\begin{aligned}
I_7 &= \int_0^\tau \int_{\mathbf{R}} \left| \frac{\bar{u}_{yy}}{\bar{\rho}^3} \phi_y \phi \right| dy d\tau \\
&\leq \frac{1}{8} \int_0^\tau \left\| \bar{\rho}^{\frac{\gamma-3}{2}} \phi_y \right\|^2 d\tau + C \frac{\epsilon}{\delta} \int_0^\tau \int_{\mathbf{R}} \bar{\rho}^{\gamma-2} \bar{u}_y \phi^2 |\bar{u}_{yy}| \bar{\rho}^{-1-2\gamma} dy d\tau \\
&\leq \frac{1}{8} \int_0^\tau \left\| \bar{\rho}^{\frac{\gamma-3}{2}} \phi_y \right\|^2 d\tau + C \frac{\epsilon^3}{\delta^3 \mu^{1+2\gamma}} \int_0^\tau \int_{\mathbf{R}} \bar{u}_y \bar{\rho}^{\gamma-2} \phi^2 dy d\tau \\
&\leq \frac{1}{8} \int_0^\tau \left\| \bar{\rho}^{\frac{\gamma-3}{2}} \phi_y \right\|^2 d\tau + \frac{1}{8} \int_0^\tau \int_{\mathbf{R}} \bar{u}_y \bar{\rho}^{\gamma-2} \phi^2 dy d\tau.
\end{aligned} \tag{3.36}$$

It follows from (3.31) that

$$\begin{aligned}
I_8 &= \int_0^\tau \int_{\mathbf{R}} \left| \frac{\bar{\rho}_{yy}}{\bar{\rho}^3} \phi_y \psi \right| dy d\tau \\
&\leq C \epsilon \delta^{-1} \int_0^\tau \int_{\mathbf{R}} |\bar{\rho}^{-\frac{3+\gamma}{2}} \bar{u}_y \phi_y \psi| dy d\tau + C \epsilon \delta^{-1} \int_0^\tau \int_{\mathbf{R}} |\bar{\rho}^{-1-\gamma} \bar{u}_y \phi_y \psi| dy d\tau \\
&\leq \frac{1}{8} \int_0^\tau \left\| \bar{\rho}^{\frac{\gamma-3}{2}} \phi_y \right\|^2 d\tau + C \epsilon^2 \delta^{-2} \int_0^\tau \int_{\mathbf{R}} (\bar{\rho}^{-2\gamma} + \bar{\rho}^{1-3\gamma}) |\bar{u}_y|^2 \psi^2 dy d\tau \\
&\leq \frac{1}{8} \int_0^\tau \left\| \bar{\rho}^{\frac{\gamma-3}{2}} \phi_y \right\|^2 d\tau + C \epsilon^3 \delta^{-3} \mu^{-3\gamma} \int_0^\tau \int_{\mathbf{R}} \bar{\rho} \bar{u}_y \psi^2 dy d\tau \\
&\leq \frac{1}{8} \int_0^\tau \left\| \bar{\rho}^{\frac{\gamma-3}{2}} \phi_y \right\|^2 d\tau + \frac{1}{8} \int_0^\tau \int_{\mathbf{R}} \bar{\rho} \bar{u}_y \psi^2 dy d\tau, \quad \text{if } \epsilon \ll 1.
\end{aligned} \tag{3.37}$$

Finally, one has

$$I_9 = \int_0^\tau \int_{\mathbf{R}} \left| g \frac{\phi_y}{\bar{\rho}^2} \right| dy d\tau \leq \frac{1}{8} \int_0^\tau \left\| \bar{\rho}^{\frac{\gamma-3}{2}} \phi_y \right\|^2 d\tau + C \int_0^\tau \int_{\mathbf{R}} \frac{g^2}{\bar{\rho}^{1+\gamma}} dy d\tau. \tag{3.38}$$

Recalling that (3.7), (3.13) and (3.29), one can get

$$\begin{aligned}
|g| &\leq |\bar{u}_{yy}| + |\bar{\rho} \bar{u}_y \psi| + C |\bar{\rho}^{\gamma-2} \bar{\rho}_y \phi| \\
&\leq |\bar{u}_{yy}| + |\bar{\rho} \bar{u}_y \psi| + C |\bar{\rho}^{\frac{\gamma-1}{2}} \bar{u}_y \phi|.
\end{aligned} \tag{3.39}$$

Thus the last term in (3.38) can be estimated by

$$\begin{aligned}
&\left| \int_0^\tau \int_{\mathbf{R}} \frac{g^2}{\bar{\rho}^{1+\gamma}} dy d\tau \right| \\
&\leq C \int_0^\tau \int_{\mathbf{R}} \frac{\bar{u}_{yy}^2}{\bar{\rho}^{1+\gamma}} dy d\tau + C \int_0^\tau \int_{\mathbf{R}} \bar{\rho}^{1-\gamma} \bar{u}_y^2 \psi^2 dy d\tau + C \int_0^\tau \int_{\mathbf{R}} \bar{\rho}^{-2} \bar{u}_y^2 \phi^2 dy d\tau \\
&\leq C \frac{\epsilon^3}{\mu^{1+\gamma}} \int_0^\tau \left\| \bar{u}_{xx} \right\|_{L^2(dx)}^2 d\tau + C \frac{\epsilon}{\delta \mu^\gamma} \int_0^\tau \int_{\mathbf{R}} (\bar{\rho} \bar{u}_y \psi^2 + \bar{\rho}^{\gamma-2} \bar{u}_y \phi^2) dy d\tau \\
&\leq C \frac{\epsilon}{\delta \mu^{1+\gamma}} \int_0^\tau \left(\tau + \frac{\delta}{\epsilon} \right)^{-2} d\tau + C \frac{\epsilon}{\delta \mu^\gamma} \int_0^\tau \int_{\mathbf{R}} (\bar{\rho} \bar{u}_y \psi^2 + \bar{u}_y \bar{\rho}^{\gamma-2} \phi^2) dy d\tau \\
&\leq C \frac{\epsilon^2}{\delta^2 \mu^{1+\gamma}} + \frac{1}{8} \int_0^\tau \int_{\mathbf{R}} (\bar{\rho} \bar{u}_y \psi^2 + \bar{u}_y \bar{\rho}^{\gamma-2} \phi^2) dy d\tau, \quad \text{if } \epsilon \ll 1.
\end{aligned} \tag{3.40}$$

Substituting (3.27)-(3.40) into (3.26) and recalling (3.12), it holds that

$$\begin{aligned} & \int_{\mathbf{R}} \left(\bar{\rho} \psi^2 + \bar{\rho}^{\gamma-2} \phi^2 + \phi_y^2 \right) dy + \int_0^\tau \int_{\mathbf{R}} \left(\psi_y^2 + \bar{\rho}^{\gamma-2} \bar{u}_y \phi^2 + \bar{\rho} \bar{u}_y \psi^2 + \bar{\rho}^{\gamma-3} \phi_y^2 \right) dy d\tau \\ & \leq C \epsilon^{(1/2-a)} |\ln \epsilon|^{-1/2}, \quad \text{if } \epsilon \ll 1. \end{aligned} \quad (3.41)$$

Step 3. As the last step, we estimate $\sup_{\tau \in [0, \tau_1]} \|\psi_y\|$. For this, multiplying (3.5) by $-\psi_{yy}/\rho$ gives

$$\left(\frac{\psi_y^2}{2} \right)_\tau - (\psi_y \psi_\tau + u \frac{\psi_y^2}{2})_y + u_y \frac{\psi_y^2}{2} - p'(\rho) \frac{\phi_y \psi_{yy}}{\rho} + \frac{\psi_{yy}^2}{\rho} = g \frac{\psi_{yy}}{\rho}. \quad (3.42)$$

Integrating the above equation over $\mathbf{R}^1 \times [0, \tau]$ yields

$$\int_{\mathbf{R}} \frac{\psi_y^2}{2} dy + \int_0^\tau \int_{\mathbf{R}} \left(\frac{\bar{u}_y \psi_y^2}{2} + \frac{\psi_{yy}^2}{\rho} \right) dy d\tau = \int_0^\tau \int_{\mathbf{R}} \left\{ p'(\rho) \frac{\psi_{yy} \phi_y}{\rho} + g \frac{\psi_{yy}}{\rho} - \frac{\psi_y^3}{2} \right\} dy d\tau. \quad (3.43)$$

First, one has

$$\left| \int_0^\tau \int_{\mathbf{R}} p'(\rho) \frac{\psi_{yy} \phi_y}{\rho} dy d\tau \right| \leq \frac{1}{8} \int_0^\tau \int_{\mathbf{R}} \frac{\psi_{yy}^2}{\rho} dy d\tau + C \int_0^\tau \|\bar{\rho}^{\frac{\gamma-3}{2}} \phi_y\|^2 d\tau. \quad (3.44)$$

Then it follows from (3.40) and (3.41) that

$$\begin{aligned} \left| \int_0^\tau \int_{\mathbf{R}} g \frac{\psi_{yy}}{\rho} dy d\tau \right| & \leq \frac{1}{8} \int_0^\tau \int_{\mathbf{R}} \frac{\psi_{yy}^2}{\rho} dy d\tau + C \left| \int_0^\tau \int_{\mathbf{R}} \frac{g^2}{\bar{\rho}} dy d\tau \right| \\ & \leq \frac{1}{8} \int_0^\tau \int_{\mathbf{R}} \frac{\psi_{yy}^2}{\rho} dy d\tau + C \epsilon^{1/2-a} |\ln \epsilon|^{-1/2}. \end{aligned} \quad (3.45)$$

Furthermore, we can compute that

$$\begin{aligned} \left| \int_0^\tau \int_{\mathbf{R}} \frac{\psi_y^3}{2} dy d\tau \right| & \leq C \int_0^\tau \|\psi_{yy}\|^{\frac{1}{2}} \|\psi_y\|^{\frac{5}{2}} d\tau \\ & \leq \frac{1}{8} \int_0^\tau \int_{\mathbf{R}} \frac{\psi_{yy}^2}{\rho} dy d\tau + C \int_0^\tau \|\psi_y\|^{\frac{10}{3}} d\tau \\ & \leq \frac{1}{8} \int_0^\tau \int_{\mathbf{R}} \frac{\psi_{yy}^2}{\rho} dy d\tau + C \sup_{\tau \in [0, \tau_1]} \|\psi_y\|^{\frac{4}{3}} \int_0^\tau \|\psi_y\|^2 d\tau \\ & \leq \frac{1}{8} \int_0^\tau \int_{\mathbf{R}} \frac{\psi_{yy}^2}{\rho} dy d\tau + C \int_0^\tau \|\psi_y\|^2 d\tau, \end{aligned} \quad (3.46)$$

where in the last inequality we have used the a priori assumption

$$\sup_{\tau \in [0, \tau_1]} \|\psi_y\| \leq 1. \quad (3.47)$$

Substituting (3.44), (3.45) and (3.46) into (3.43) and using (3.13) and (3.41), it holds that

$$\int_{\mathbf{R}} \psi_y^2 dy + \int_0^\tau \int_{\mathbf{R}} \left(\bar{u}_y \psi_y^2 + \frac{\psi_{yy}^2}{\bar{\rho}} \right) dy d\tau \leq C \epsilon^{(1/2-a)} |\ln \epsilon|^{-1/2}. \quad (3.48)$$

Therefore, (3.17) can be derived directly from (3.41) and (3.48) and the a priori assumption (3.47) is verified if ϵ is suitably small. It follows from (3.17) that if $1 < \gamma \leq 2$, then

$$\begin{aligned} \sup_{0 \leq \tau \leq \tau_1} \|\phi(\cdot, \tau)\|_{L^\infty} &\leq \sqrt{2} \sup_{0 \leq \tau \leq \tau_1} \|\phi(\cdot, \tau)\|^{1/2} \|\phi_y(\cdot, \tau)\|^{1/2} \\ &\leq C \sup_{0 \leq \tau \leq \tau_1} \left(\int_{\mathbf{R}} \bar{\rho}^{\gamma-2} \phi^2 dy \right)^{1/4} \left(\int_{\mathbf{R}} \phi_y^2 dy \right)^{1/4} \\ &\leq C \epsilon^{1/6} |\ln \epsilon|^{-1/4}, \end{aligned} \quad (3.49)$$

and if $\gamma > 2$, then

$$\begin{aligned} \sup_{0 \leq \tau \leq \tau_1} \|\phi(\cdot, \tau)\|_{L^\infty} &\leq \sqrt{2} \sup_{0 \leq \tau \leq \tau_1} \|\phi(\cdot, \tau)\|^{1/2} \|\phi_y(\cdot, \tau)\|^{1/2} \\ &\leq C \sup_{0 \leq \tau \leq \tau_1} \left(\int_{\mathbf{R}} \mu^{2-\gamma} \bar{\rho}^{\gamma-2} \phi^2 dy \right)^{1/4} \left(\int_{\mathbf{R}} \phi_y^2 dy \right)^{1/4} \\ &\leq C \mu^{\frac{1}{2}-\frac{\gamma}{4}} \epsilon^{\frac{1}{4}-\frac{1}{2(\gamma+4)}} |\ln \epsilon|^{-\frac{1}{4}} \\ &\leq C \epsilon^{\frac{1}{\gamma+4}} |\ln \epsilon|^{\frac{1-\gamma}{4}}. \end{aligned} \quad (3.50)$$

And we also have

$$\sup_{\tau \in [0, \tau_1]} \|\psi_y\| \leq C \epsilon^{(\frac{1}{4}-\frac{a}{2})} |\ln \epsilon|^{-1/4} \leq 1. \quad (3.51)$$

So from (3.49)-(3.51) the a priori assumption (3.10) is verified if $\epsilon \ll 1$. On the other hand, by using Sobolev inequality, one can get

$$\begin{aligned} \sup_{0 \leq \tau \leq \tau_1} \|\psi(\cdot, \tau)\|_{L^\infty} &\leq \sqrt{2} \sup_{0 \leq \tau \leq \tau_1} \|\psi(\cdot, \tau)\|^{1/2} \|\psi_y(\cdot, \tau)\|^{1/2} \\ &\leq \sqrt{2} \mu^{-\frac{1}{4}} \sup_{0 \leq \tau \leq \tau_1} \|\sqrt{\bar{\rho}} \psi(\cdot, \tau)\|^{1/2} \|\psi_y(\cdot, \tau)\|^{1/2} \\ &\leq \begin{cases} C \epsilon^{1/8} |\ln \epsilon|^{-1/2}, & \text{if } 1 < \gamma \leq 2, \\ C \epsilon^{\frac{\gamma+1}{4(\gamma+4)}} |\ln \epsilon|^{-1/2}, & \text{if } \gamma > 2, \end{cases} \end{aligned} \quad (3.52)$$

Thus the convergence rate (3.18) is justified and the proof of Lemma 3.2 is completed. \square

Proof of Theorem 1.1: It remains to prove (1.7) with a, b given in (1.8) and (1.9) respectively. From Lemma 2.2, Lemma 2.3 (iii) and Theorem 3.1 and recalling that

$\mu = \epsilon^a |\ln \epsilon|$, $\delta = \epsilon^a$, it holds that for any given positive constant h , there exist a constant $C_h > 0$ which is independent of ϵ such that

$$\begin{aligned}
& \sup_{t \geq h} \|\rho(\cdot, t) - \rho^r(\frac{\cdot}{t})\|_{L^\infty} \\
& \leq \sup_{0 \leq \tau \leq +\infty} \|\phi(\cdot, \tau)\|_{L^\infty} + \sup_{t \geq h} \|\bar{\rho}(\cdot, t) - \rho_\mu^r(\frac{\cdot}{t})\|_{L^\infty} + \sup_{t \geq h} \|\rho_\mu^r(\frac{\cdot}{t}) - \rho^r(\frac{\cdot}{t})\|_{L^\infty} \\
& \leq \begin{cases} C_h \left(\epsilon^{1/6} |\ln \epsilon|^{-1/4} + \delta |\ln \delta| + \mu \right), & \text{if } 1 < \gamma \leq 2, \\ C_h \left(\epsilon^{\frac{1}{\gamma+4}} |\ln \epsilon|^{(1-\gamma)/4} + \delta |\ln \delta| + \mu \right), & \text{if } \gamma > 2, \end{cases} \\
& \leq C_h \epsilon^a |\ln \epsilon|,
\end{aligned}$$

and

$$\begin{aligned}
& \sup_{t \geq h} \|m(\cdot, t) - m^r(\frac{\cdot}{t})\|_{L^\infty} \\
& \leq \sup_{t \geq h} \left(\|m(\cdot, t) - \bar{m}(\cdot, t)\|_{L^\infty} + \|\bar{m}(\cdot, t) - m_\mu^r(\frac{\cdot}{t})\|_{L^\infty} + \|m_\mu^r(\frac{\cdot}{t}) - m^r(\frac{\cdot}{t})\|_{L^\infty} \right) \\
& \leq C \sup_{0 \leq \tau \leq +\infty} \left(\|\psi(\cdot, \tau)\|_{L^\infty} + \|\phi(\cdot, \tau)\|_{L^\infty} \right) \\
& \quad + \sup_{t \geq h} \left(\|\bar{m}(\cdot, t) - m_\mu^r(\frac{\cdot}{t})\|_{L^\infty} + \|m_\mu^r(\frac{\cdot}{t}) - m^r(\frac{\cdot}{t})\|_{L^\infty} \right) \\
& \leq \begin{cases} C_h \left(\epsilon^{1/8} |\ln \epsilon|^{-1/2} + \epsilon^{1/6} |\ln \epsilon|^{-1/4} + \delta |\ln \delta| + \mu \right), & \text{if } 1 < \gamma \leq 2, \\ C_h \left(\epsilon^{\frac{\gamma+1}{4(\gamma+4)}} |\ln \epsilon|^{-1/2} + \epsilon^{\frac{1}{\gamma+4}} |\ln \epsilon|^{(1-\gamma)/4} + \delta |\ln \delta| + \mu \right), & \text{if } \gamma > 2, \end{cases} \\
& \leq \begin{cases} C_h \epsilon^b |\ln \epsilon|^{-\frac{1}{2}}, & \text{if } 1 < \gamma < 3, \\ C_h \epsilon^{\frac{1}{\gamma+4}} |\ln \epsilon|, & \text{if } \gamma \geq 3, \end{cases}
\end{aligned}$$

Thus the proof of Theorem 1.1 is completed. \square

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